Lecture of Awardee 3

From Thinking in Action to Mathematical Models — A View from Developmental Psychology

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ABSTRACT Developmental psychologists agree that intelligent action precedes language in children's development and that language transforms children's thinking. In this lecture I explore the ways in which children's thinking in action is transformed by learning to use conventional mathematical signs to represent quantities and relations between quantities. Numbers have two types of meaning: a referential meaning, which connects numbers to quantities, and an analytical meaning, which is intrinsic to the conventional systems of signs. This dual nature of numbers means that, from the psychological perspective, numbers are models of the world. The referential meaning of numbers is based on children's use of schemas of action to establish relations between quantities; it is at the core of quantitative reasoning. The analytical meaning rests of the rules that define relations between numbers in a conventional system and provides the basis for arithmetic. This paper presents research that illustrates how teaching can build a bridge between thinking in action and mathematical models by promoting the coordination of quantitative reasoning with number knowledge.

Keywords: Quantitative reasoning; Multiplicative reasoning; Action schemas; Reasoning in action; Referential meaning of number; Analytical meaning of number; Cultural systems of signs.

This paper starts from what I take to be an uncontroversial idea: mathematical modelling is a form of intelligent action. Although this idea might seem trivial, it has profound implications for thinking about how mathematics is learned and used to model the world. By examining this idea from the perspective of developmental psychology, I consider here four fundamental questions:

1. What is the origin of intelligence?
2. How does the learning of numerical signs change children's thinking about quantities?
3. What are the basic types of relations between quantities that students need to master in primary school?
4. How can schools promote students' thinking about relations between quantities?

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These four questions are examined in the context of mathematics teaching in school, but it is important to stress that mathematical reasoning develops also outside school (Nunes et al., 1993). However, an analysis of what and how people develop mathematical reasoning outside school is beyond the scope of this paper.

1. The Origin of Intelligence

For the fathers of the study of human intelligence, there was no doubt that intelligence starts to develop before language; this means that language cannot be the origin of intelligence. Gesthalt psychologists, such Wertheimer and Köhler, as well as child psychologists, such as Binet and Piaget, studied intelligent action in subjects who did not have language; Gesthalt psychologists studied monkeys and developmental psychologists studies human babies. The paradigms that were used to investigate intelligent action in the absence of language had in common the fact that the subjects could not attain the aim of their action directly: they had to use inferences that connected different aspects of the world, and by doing so they were able to attain their goal. These classical paradigms can be illustrated by detour problems and the use of tools.

In detour problems, the subject seeks to obtain an object, but is separated from the object by a barrier, as in Figure 1. Because the direct movement towards the object is obstructed, the subject must conceive of two movements as a single path: the first movement, A, away from the object, is reversed by the second one, B, towards the object, which is located in position C. Such behavior exemplifies a practical inference, which puts two different pieces of information together and leads to the conclusion of an equivalence: the movement from A to C is equivalent to the sum of the movements from A to B plus B to C.

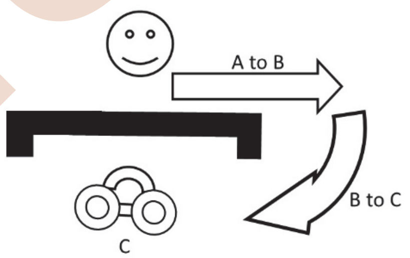

Fig. 1. The baby tries to reach the toy over a barrier but does not succeed. By thinking of a movement away from the toy as cancelled by another one, the baby can reach the toy.

A second classical paradigm in the study of intelligent action in the absence of language involves the use of tools: the subject tries to reach an object directly but the object is out of reach; by using a tool, such as a stick, the subject can move the object into reach. The use of tools also illustrates a practical inference: object A cannot be reached by the subject, but the subject can reach object B which in turn can reach object A, and B can be used to bring A within reach.

A large number of studies using the detour and tool use paradigms leaves no doubt that monkeys and babies can solve these problems and thus show intelligent action in the absence of language. Alternative theories were proposed early on in psychology to explain how detour and tool use problems were solved. The two initial hypotheses were proposed by Gesthalt psychologists and learning theorists.

Gesthalt psychologists suggested that the solution of detour and tool use problems involved a sudden re-organization of the perception of the situation, termed insight. Thus they argued that intelligence was rooted in perception, but there was no explanation for what led to this sudden perceptual re-organization. Insight provided a structure for further intelligent action: the subject became able to use previous insights to solve new problems that required the same sort of inference.

Learning theorists suggested that solution was accomplished by trial and error, also known as the law of effect, which involves a gradual approach to the solution based on the reinforcement of those actions followed by success. Such learning had no structuring role on further problem solving because learning was viewed as essentially guided by reinforcement, which is external to the learner.

Piaget criticized both theories. He argued that the Gesthaltists offered a description of structures for thinking, but no explanation for their origin, whereas the trial and error theory sought to explain the origin of the solution but provided no description of structures for thinking. Piaget proposed that the solution of problems in action was accomplished by learning to coordinate initially independent actions into structured sequences, that are repeatable and useful in different situations. Such sequences he termed schemas of action. According to Vergnaud (2009), action schemas contain theorems in action, i.e. involve a process of making inferences which are not explicitly recognized by the actor.

In summary, the existence of intelligent action in the absence of language is accepted wisdom in psychology: intelligence starts in action. Although the description of the processes by which problems are solved in action differs across theories, all theories about the development of human intelligence include a period during which babies are able to make practical inferences and to solve problems, even though they do not use language to communicate and cannot use language to represent the process of making inferences nor to represent the solution. It is also accepted wisdom in psychology that the acquisition of conventional systems of signs, such as language and number systems, changes the possibilities of human intelligent action. This idea is explored in the next section.

2. How Does the Learning of Numerical Signs Change Children's Thinking about Quantities?

All theories of cognitive development propose that the acquisition of signs transforms intelligence. Consider the case of numerical signs. Human babies can compare small quantities and distinguish two from three objects, for example; they can also compare larger quantities when the difference between the quantities is large, such as 6 versus 20 objects. But they do not succeed in distinguishing, for example, 19 from 20 objects visually, and neither do adults. Because babies do not know how to count, their ability to compare discrete quantities is restricted to what they can distinguish perceptually. Older children and adults, who are not restricted to visual comparisons and can use a counting system to compare discrete quantities, can easily tell the difference between a set with 19 and one with 20 items. They are no longer limited by their perceptual skills. Learning to count changes human's ability to compare sets.

Learning to count does not immediately transform children's ability to compare discrete quantities. Children learn how to count, but continue to try to compare quantities just by looking at them until about the age of 6 or 7 years. Many 4- and 5-year old children who can count objects up to ten do not count to compare sets with seven or eight items, for example, and do not succeed in making accurate comparisons when the perceptual arrangement of the items makes visual comparison difficult. This observation has been replicated in several countries with different counting and educational systems, such as Brazil, China, England, France, Switzerland and the US (for a review, see Nunes & Bryant, 2022 a).

The explanation for this phenomenon is a matter of controversy, which in my view boils down to the theory of number meanings upon which the different explanations are based. In this paper, I contrast two radically different explanations for how children learn the meanings of words, and in particular the meaning of number words. The general terms for these two basic explanations, associationism and representational theories, cover a variety of specific approaches that differ in their details, but still allows the theory to be classified as belonging to one of these two types. This discussion does not focus on the variations as its aim is to contrast the two general theoretical approaches.

The associationist view

According to the associationist view, which is considered the oldest theory of thought (for a review, see Mandelbaum, 2015), a word attains its meaning by being heard at the same time as its referent is perceived; i.e. by association based on contiguity in time and place, to use Hume's (1896) expression. In the same way, a number word attains its meaning by association to a perceived numerosity: the word "three", for example, becomes associated with the perceived numerosity of a group with three items.

Among the many criticisms of this view of how number words acquire meaning, I consider one the most crucial: in this theory, there is no principled connection between the meaning of the words "three" and "four", for example: each of these words acquires its meaning by association to a referent, which is a set with a specific numerosity. This problem of associationism as a theory of number meanings has been recognized by researchers, who tried to solve it by resorting to other, different processes which would be complementary to association. Carey (2004), for example, suggested that children learn the meaning of the number words “one” and “two” by association, but at the same time they distinguish each of the two items in the set with the numerosity of two. This process, termed parallel individuation, eventually allowed the child to realize that “two” means one more than one, and that “three” means one more than two. These realizations eventually would lead a child who is learning to count to generalize from the initial sequences “one, two, three” to all sequences in counting, and to infer that that each number word refers to a set that has one more item than the set represented by the number word that precedes it in the sequence. Thus, in Carey’s theory, the meaning of numbers ceases to be the result of associations between words and referents and comes to rely on a new process.

Carey’s theory ends up by breaking with associationism. To quote her conclusion: “We cannot just teach our children to count and expect that they will then know what 'two' or 'five' means. Learning such words, even without fully understanding them, creates a new structure, a structure that can then be filled in by mapping relations between these novel words and other, familiar concepts” (Carey, 2004, p 68). Even though Carey does not clarify what she means by the other, familiar concepts, her explanation departs from associationism in so far as it depends on internal conceptual structures provided by the child rather than on association. It is worth noting that Carey explicitly relies on numerical comparisons as part of the internal structures when she describes the child’s realization that two is one more than one. In order for this explanation to be in line with findings in mathematics education, numerical comparisons would have to be an early achievement, but research in mathematics education has shown that the mastery of numerical comparisons is a late achievement, rather than an early one (e.g. Carpenter et al., 1981; 1982; Hudson, 1983).

The representational view

The representational view offers an alternative to associationism and it is also a classical theory in psychology (von Helmholtz, 1921). I present here the synthesis to which we (Nunes & Bryant, 2015 a; 2022 a; 2022 b) have arrived over the years. We consider numbers to be elements of a system of signs and to have two meanings.

The first meaning of numbers is representational, extrinsic to the number system, and connects numbers to quantities through measurement. According to Thompson (1993), a person constitutes a quantity when he/she thinks of a quality as susceptible of measurement. Measurement is the attribution of numbers to quantities according to rules (Stevens, 1946) that represent relations between quantities: thus, in the representational view, numbers are relational concepts.

When we measure extensive quantities, the measure is obtained by the addition of its units. If the quantity is discrete, the unit is often a natural unit: e.g. oranges, rabbits, pencils etc. When the child conceives of counting as the addition of each unit, as it is counted, to a set of already counted items (von Helmholtz, 1921), the child has a first insight into the representational meaning of natural numbers. The action schema of joining and its inverse, separating, are the sources for the representational meaning of natural numbers. If the quantity is continuous, a conventional unit is defined, and the total quantity is conceived as the sum of all the conventional units, such as centimeters or inches. The logic of additive relations between the units in a quantity supports counting systems in different languages, although the number words and the particular organization (e.g. base ten, mixed base ten and twenty, base 12) of counting systems differs. Understanding the connection between number and continuous quantities is a later achievement, but the logic of additive relations applies also to continuous quantities.

Measuring intensive quantities is yet a later achievement. Intensive quantities are measured by the ratio between two different extensive quantities (Tolman, 1917): e.g. speed is measured by $\mathrm{km/hour}$ and density is measured by mass/volume. However, a discussion of intensive quantities and rational numbers is beyond the scope of this paper, which focuses on natural numbers (for further discussion, see Nunes et al., 2022b).

In a representational theory, number meanings are not based on the association between words and referents, but rather on the ability to think logically about quantities and to make logical inferences on the basis of relations between quantities. Thus numbers are relational concepts that represent relations between the units within a quantity being measured; natural numbers can be used to determine the numerosity of groups of items, but their meaning does not result from the associations with numerosities.

The second meaning of numbers is formal, analytical, intrinsic to the system and defined by the rules of the system. Natural number systems use addition rules to define numbers: for example, 9 means $8 + 1$ because we add 1 to 8 to get to 9. In fact, any number has infinite analytical meanings: 9 means $8 + 1$, $7 + 2$, $6 + 3$, $10 - 1$, $11 - 2$ and so on. The rule of addition justifies the cardinal meaning of numbers: one can say that a set has, for example, 9 objects, because as each object is counted, it is added to the set of already counted items. Addition also justifies the ordinal meaning of numbers: 9 is more than 8 (and all its predecessors) because one gets to 9 by adding 1 to 8. In contrast, rational numbers are not based on addition, but on ratios, and thus lead to a rather different system of analytical meanings for numbers (for further discussion, see Nunes & Bryant, 2022 b).

Learning numerical signs and grasping their analytical meanings enables children to think about quantities in a new way. Whereas thinking in action depends on the quantities being present and on the child's ability to manipulate items, thinking that uses numerical representations is freed from this restriction. Steffe (1992) suggests that a simple example of this capacity to use numbers to think about quantities can be found when children are able to answer a problem such as: "There are seven marbles inside this cup; I will put four marble in; how many marbles will the in the cup?". The researcher puts the marbles in the cup as the child watches. If the child counts on from seven, saying "eight, nine, ten, eleven", the child has demonstrated that number words (in this case, seven) can now stand for a whole collection of objects; the seven marbles do not have to be seen to become part of a larger collection. Another example of this empowerment by the use of symbols is manifested in counting money: when a child can point to a single coin that has the value of 10 pence and count on from 10, in order to pay for example 13 pence using a combination of 10p and 1p coins, the child's thinking has been empowered by the use of symbols (Nunes et al., 2015 b). Neither of these tasks, in which a number word has to stand for a measure, is mastered by children immediately as they learn to count, but only later on, when they grasp the role of addition in counting.

One further example: If we tell a child that there are 12 objects in a box and 8 in another box, the child who understands the relational meaning of number words no longer needs to compare the quantities directly by looking at them and knows in which box there are more objects. Once again, this achievement cannot be taken for granted when a child has learned how to count: Davidson et al (2012) showed that children who could count to, for example, 20 were not necessarily able to make comparisons between groups of objects inside boxes on the basis of their numerical labels. Thus, learning to count is necessary but not sufficient for grasping the analytical meaning of numbers: the analytical meaning depends on understanding the relational meaning of each number, defined as an addition of one to the previous number in a counting system.

Number meanings and mathematical skills

Each of these two types of number meaning is associated with a different mathematical skill: the representational meaning is related to quantitative reasoning and the analytical meaning is related to arithmetic (Nunes et al., 2016). Quantitative reasoning is the ability to make logical inferences on the basis of relations between quantities. One can reason about quantities without representing them with numbers: for example, if you know that I have red pencils and blue pencils, you can infer that the total number of pencils I have is greater than the number of either blue or red pencils, and that the number of red pencils is equal to the total number of pencils minus the number of blue pencils. You don't have to know the number of pencils to think about these relations, which are additive (i.e. based on addition and subtraction) and relate to part-whole reasoning. Thompson (1993) suggested that numbers are not relevant to quantitative reasoning; in fact, quantitative reasoning is the source for the representational meaning of numbers.

In contrast, arithmetic skill is the ability to analyze "the behavior of various numbers in four operations: addition, subtraction, multiplication, and division" (Guedj, 1998). Arithmetic skill is related to the analytical meaning of numbers. When students become able to coordinate quantitative reasoning and arithmetic, they are able to use mathematical models to understand the world; this is a radically new step beyond thinking in action, but originates from action schemas. The remainder of this paper does not focus on the analytical meaning of numbers and arithmetic, but on quantitative reasoning and on the relations between quantities that students need to master in primary school.

Quantitative reasoning and arithmetic are correlated, but distinct abilities, both theoretically and empirically, because each makes an independent contribution to the prediction of mathematics achievement, even after controlling for general cognitive ability (Nunes et al., 2012). Unfortunately, these two abilities have not been clearly distinguished in school nor in research: for example, word problems are designed to test students' skills in applying arithmetic to specific situations, but the focus is more often on the arithmetic operations that students have just learned to calculate than on the relations between quantities described in the word problems. With notable exceptions (e.g. Cheong et al., 2002; Kho et al., 2014; Ng et al., 2009; Nunes et al., 2015b; Thompson, 1993), teaching and research has been more concerned with arithmetic than with quantitative reasoning.

In summary, there are two different types of theory about how learning numerical signs changes intelligence in action. I suggest that the difference between the theories boils down to their explanations for how people learn number meanings. According to associationism, number signs acquire meaning by being associated with particular numerosities. In associationism, learning number signs doesn't actually change intelligent action because there is no principled connection between number words; this is a fundamental weakness of associationims. Attempts to deal with this weakness (e.g. Carey, 2004) led to a move away from associationism, because such attempts introduced the idea of change in the child's thinking structures, but the idea of thinking structures is incompatible with associationism. In our view (Nunes & Bryant, 2022 a and b), a representational theory of number meanings provides an alternative to associationism and a good description of how signs change intelligent action. When children realize that, as they count (i.e. use number signs) they are adding objects to the already counted group, they can explore the consequences of the action schemas of joining, separating and setting in one-to-one correspondence for relations between numbers of objects in collections.

Two distinct, even though correlated, mathematical abilities are related to each of the two meanings of number: quantitative reasoning and arithmetic. Because arithmetic has maintained such a prominent role in school mathematics, the teaching of quantitative reasoning has received comparatively little attention. In the two final sections of this paper, the focus is on quantitative reasoning and possibilities as well as obstacles to its teaching in school.

3. What are the Basic Types of Relations between Quantities that Students Need to Master in Primary School?

In order to think more about quantitative reasoning, it is useful to start from a classification of the types of relations between quantities that students need to master in primary school. Previous classifications by different researchers (e.g. Harel et al., 1994; Nesher, 1988; Vergnaud, 1983) have distinguished between additive and multiplicative reasoning. The distinction between these two types of relations between quantities becomes crystal clear when one considers the action schemas as well as the logical relations involved in understanding additive and multiplicative relations. Because these are the two types of relation between quantities that are crucial for learning mathematics in the first eight years in school, it is vital that research and teaching identify and promote each of these forms of reasoning. However, as Thompson et al (2003) pointed out, there is still a school of thought that considers multiplication as repeated addition, an approach to teaching multiplication that seems to rest on the assumption that children's intuitions about multiplication is inevitably grounded in repeated addition. Teaching multiplicative reasoning as repeated addition obscures the difference between the two types of relation between quantities.

In our view (Nunes et al., 2015 a), additive reasoning is grounded in the logic of part-whole relations between quantities, which in turn is anchored in the action schemas of joining, separating, and setting items in one-to-one correspondence. Multiplicative reasoning is grounded in the logic of a fixed ratio between quantities;

Additive reasoning

- Together Rob and Ann have 15 books (a sentence about a quantity).
- Ann has 5 books fewer than Rob (a sentence about a relation, which can be rephrased as Rob has 5 more books than Ann).
- How many books does each one have?

A diagram that shows part-whole relations between quantities

Multiplicative reasoning

- Together Rob and Ann have 15 books (a sentence about a quantity)
- Rob has twice as many books as Ann (a sentence about a relation, which can be rephrased as Ann has half the number of books that Rob has).
- Together they have 15 books.
- How many books does each one have?

A diagram that shows a fixed ratio relation between quantities

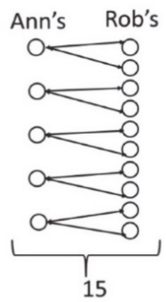

Fig. 2. The quantities in these problems are represented by the same numbers, but the relations between quantities are different. A part-whole representation of the multiplicative reasoning is possible, but conceals the fixed ratio between the two quantities

the action schema that provides a source for understanding ratio is setting items in one-to-many correspondence. Fig. 2 contrasts additive and multiplicative relations in problems that use the same numbers and that might seem rather similar to students, but can be solved by means of different action schemas. The diagrams used in the figure help to sharpen the contrast between the two types of relation between quantities and also illustrate how the use of a part-whole diagram is inappropriate to represent fixed ratios between quantities.

Much research has shown that 5 and 6-year old children can solve additive reasoning problems before school by using joining and separating schemas; comparison problems, which require setting items from two collections in one-to-one correspondence, are solved later (e.g. Carpenter et al, 1982; Ryley et al., 1983). Research also shows that from about the same age, children can solve multiplicative reasoning problems using the action schema of one-to-many correspondence before they have been taught about the operations of multiplication and division in school (e.g. Becker, 1993; Corea et al., 1998; Frydman et al., 1988; 1994; Kouba, 1989; Kornilaki et al., 2005; Mulligan, 1992; Mulligan et al, 1997; 2009; Nunes et al., 2008; 2010; Park et al., 2001).

Studies that analyzed children's competence in solving multiplicative reasoning problems have shown that it is critical that problem presentation fosters the use of the one-to-many correspondence schema by offering the children different types of material to represent each of the two quantities that have to be set in a fixed ratio (e.g. Ellis, 2015; Nunes et al., 2010; Piaget, 1952). In one study (Nunes et al., 2008b), children were asked to solve multiplicative reasoning problems of different types (multiplication, sharing and division in quotas) with different unit ratios (e.g. 1:2; 1:3; 1:6) using manipulatives. At the time the study was carried out, schools in England did not include the teaching of multiplicative reasoning in the curriculum until Grade 3; the children participated in this study before this teaching. The children solved three sets of problems under different problem solving conditions. For example, in one problem the children were told a boy had 2 fishbowls and that he could fit 12 tadpoles in each bowl; the task was to figure out how many tadpoles he could have altogether. In one problem solving condition, the children had different materials to represent each of the quantities mentioned in the problem: they had cut-out circles to represent the fishbowls and blocks to represent the tadpoles. In the second condition, they had materials to represent the quantity that was a unit in the ratio: in this example, they had circles to represent the fishbowls and had to imagine the tadpoles. In the third condition, they had materials to represent the second quantity in the ratio: in this example, they had blocks to represent the tadpoles and had to imagine the fishbowls.

The problems were systematically paired with each problem solving condition across children to control for problem difficulty; the order of problem solving condition varied systematically across children to control for practice effects. Fig. 3 presents the difference in mean accuracy as a function of problem solving condition.

The mean accuracy differed across the conditions: the children performed best when they had different materials to represent each of the quantities, even though they had not received instruction in solving multiplicative reasoning problems in school. This result was replicated subsequently by Ellis (2015) with younger children in their first year in school, after multiplicative reasoning problems with the support of materials became part of the English National Curriculum.

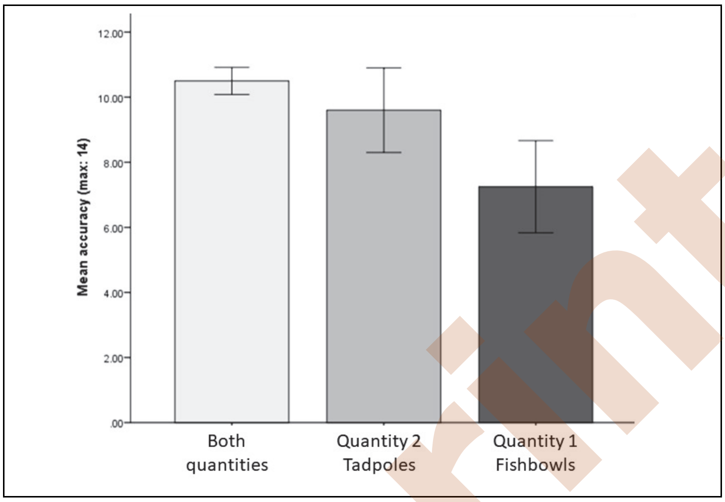

Fig. 3. Mean number of correct responses to multiplicative reasoning problems as a function of type of material that was offered to the participants to support their reasoning. Error bars represent 95% confidence interval.

In summary, even before learning about arithmetic operations in school, children can reason about relations between quantities in action. It seems common practice to encourage young children in pre-school and at the start of primary school to solve simple additive reasoning problems, but quantitative reasoning about multiplicative relations often receives little if any attention in the early years. It is possible that the belief that multiplicative reasoning stems from repeated addition is at the root of the negligence of such an important action schema in the education of young children, because it is expected that children require much practice with addition before they can start to think about multiplication. It is also possible that previous lack of awareness of the need to offer children different materials to represent the different quantities in a ratio led to the belief that young children cannot reason multiplicatively. Independently of the explanation for the neglect of teaching multiplicative reasoning in the early years, there is now plenty of research to show why it is important to include problems that involve part-whole and ratio relations in young children's mathematics education and how this can be done. I now turn to how schools can promote the use of action schemas to foster quantitative reasoning in primary school.

4. How can Schools Promote Children's Thinking about Relations between Quantities?

Even though research has shown that it is possible to cultivate students' quantitative reasoning from the time they start primary school (Nunes et al., 2007) and that this improves their learning of mathematics, quantitative reasoning has not been a traditional focus of teaching in schools (Thompson, 2011). It is now recognized that traditional mathematics curricula do not necessarily promote quantitative reasoning (Agustin, 2012; Gläser et al., 2015) and that there is an important place for quantitative reasoning in mathematics and science education (Elrod, 2014; Mayes, 2019; Panorkou et al., 2021; Smith et al., 2007). It is therefore crucial that teachers find methods to implement this new practice across the years in primary school. In this paper, some critical pointers to effective practice are presented briefly; Nunes et al (2022a) reviewed research that documented the effectiveness of these pathways to promoting quantitative reasoning.

Encouraging the use of action schemas to solve problems: Young students aged 5-7 years can solve a variety of quantitative reasoning problems in action. Teachers can encourage students to use action schemas to solve problems by providing appropriate representations that can be manipulated (Fig. 3 for the analysis of manipulatives used in multiplicative reasoning problems). When students do not seem to know how to start, teachers can suggest a starting point: for example, they can suggest "pretend these are the … in the story" or ask "show me what happened in this story: how did it start?"

Schemas of action are characterized by being applicable in a variety of situations; thus teachers can rely on manipulatives (e.g. cut-out circles, rectangles, blocks, tokens) as representations for different objects (e.g. fishbowls, cars, children, balloons) mentioned in different problems. Students become more confident when they use the same action schema, such as one-to-many correspondence, in the context of different activities (e.g. arranging children in cars to go to the zoo; placing balloons in correspondence with children who are going to a party; paying the same amount of money for each chocolate bar; sharing pencils fairly among children; figuring out how many children can sit in a hall where there is a fixed number of chairs around a table). They have the opportunity to explore the action schema from different perspectives and to start to represent it in words, which leads to a greater awareness of the action schema's organization and utility.

Reflecting about the consequences of actions in situations and using the action schemas forwards and backwards: Joining and separating are schemas of action; addition and subtraction are arithmetic operations. In order to take the step from thinking through action schemas to using thinking based on arithmetic operations, students need to understand addition as the inverse of subtraction and vice versa. The research literature has documented the difficulty of thinking of operations as the inverse of each other in problem solving (e.g. Brown, 1981; Booth, 1981) and it has also shown that there are effective approaches to support students to reflect about the inverse relations between operations (e.g. Fong et al., 2009; Nunes et al. 2008; 2009; Schneider et al., 2009; Squire et al., 2003).

Reflection about solutions obtained through action is facilitated by discussion, which requires the students to describe what they did in language and to articulate their thinking in words. Metacognition, i.e. thinking about one's own thinking, plays an important role in promoting students' thinking about relations between quantities (e.g. Garofalo et al., 1985; Kramarski et al., 2002; Lamon, 1993; Lee et al., 2014; Schoenfeld, 1987).

From actions to drawings and diagrams: It is vital that schools promote thinking in action and it is just as essential that they help students to take the step from active to iconic (i.e. using drawings and diagrams) representation and then to symbolic representations. The step from action to drawing might seem very small to an adult, but it has an impact on students' problem solving performance. In one study carried out in London schools (Kornilaki, 1999), 5- and 6- year old children were shown rectangles that represented hutches and told that there 4 rabbits in each hutch; the students' task was to take the right number of food pellets (represented by tokens) from a box so that each rabbit could have one food pellet. Students' performance was quite good: 68% of the 5-year olds and all the 6-year olds took out the right number of tokens from the box. In another study carried out in the same schools (Watanabe et al., 2000), 6 and 7-year olds were presented with a similar problem, but this time there were no manipulatives: the students were presented the problem by means of drawings and asked to present their answers in drawing. Fig. 4 shows the item presentation and a students' response.

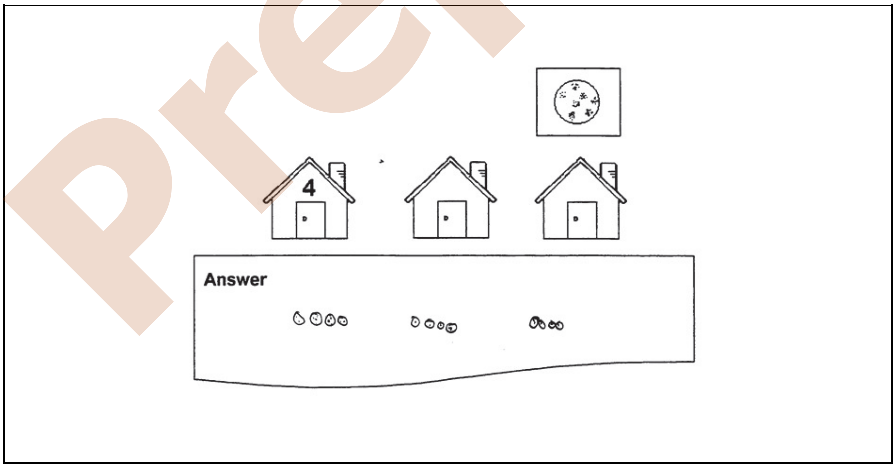

Fig. 4. The problem: In each house live 4 rabbits. They like carrot biscuits, like the one in the top box. Draw the right number of carrot biscuits so that each rabbit can have one biscuit.

In spite of the similarity in the problem, in the task presented by means of drawing the students' performance was much lower than in the task with manipulatives: 52%

of the 6-year olds and 68% of the 7-year olds answer correctly. Thus it is important that schools support students in the transition from solving problems in action to solving problems using drawings and diagrams.

Drawings and diagrams are arguably a means towards abstractions of relations between quantities (Gravemeijer, 1997) and have been used widely to support quantitative reasoning in problem solving (e.g. Brooks, 2009; Csíkos et al., 2012; Gravemeijer et al, 1990; Streefland, 1997). It is a central hypothesis in the Singapore Model Method Ang, 2001;2015; Kho et al., 2014; Ng, 2009; Yoong et al., 2009) and of the Tape Method used in Japan (Murata, 2008; She et al., 2022). There is much research about the use of drawings and diagrams, but relatively little emphasis on the distance between solving problems in action and using drawings and diagrams.

Drawings and diagrams can be used to extend students' reasoning to novel and rather problems. Cartesian or product of measures problems have been found to be significantly more difficult than the type of multiplicative reasoning problems presented earlier on in the paper (Brown, 1981; Vergnaud, 1982). However, they can be solved by the same action schema of one-to-many correspondence with some support. Figure 5 shows an example of a diagram used to encourage students aged 10–11 years to extend their multiplicative reasoning to Cartesian problems (left). Students who participated in this program initially had cut-out shapes to solve Cartesian problems and were later encouraged to use drawings and diagrams. At the end of the program, they used the same approach to describe the sample space in probability problems (Nunes et al., 2014).

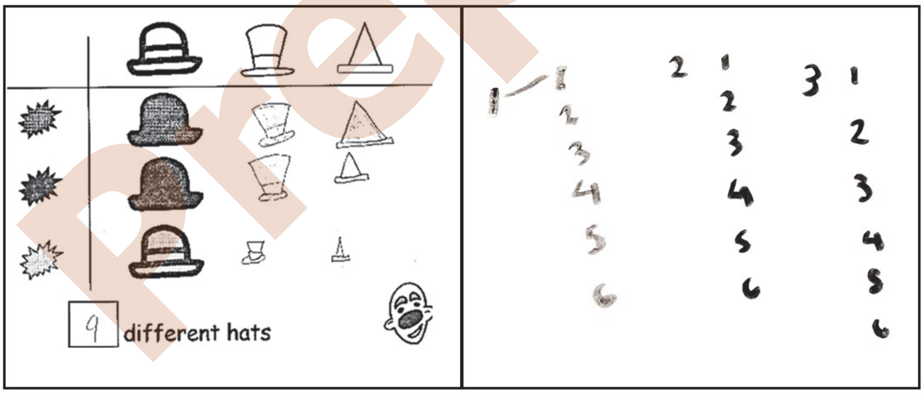

Fig. 5. Left - the problem and a student's production: A shop that sells hats lets the client choose from three different materials and three different shapes of hats. How many options does the clown have to choose from? Right – the problem and a student's production: When you throw two dice and add the numbers, is there one total that is more likely to come up than all the others?

5. Summary and Conclusions

This paper starts from the recognition that mathematical modeling is a form of intelligent action grounded in cultural mathematical practices. The aim of the paper is to explore what this assumption means for learning mathematical modeling from the perspective of developmental psychology. This aim was pursued by considering four fundamental questions. In this final section, I address these questions briefly, without contrasting the theoretical options adopted here with other possible views.

1. What is the origin of intelligence?

The origin of intelligence is in action. Psychological research about intelligent action has documented problem solving by babies before they have learned language. There is little controversy in psychology about the idea that intelligence precedes language. Piaget studied in detail how instincts and reflexes change as they are used and give rise to intelligent action. He adopted the concept of action schema to describe actions that are organized and applicable in a variety of situations and to a variety of objects, allowing the child to think in terms of relations between classes of objects and positions in space. Relational thinking is the essence of intelligent action.

2. How does the learning of numerical signs change children's thinking about quantities?

What happens to children's thinking when language is learned has proven to be hugely controversial amongst psychologists. The controversy starts with how children learn the meanings of words. In this paper, the theory adopted is the one held by eminent developmental psychologists, such as Piaget and Vygotsky, which proposes that word meanings are concepts rather than associations between two stimuli, words and objects. Therefore, children's conceptual development is intimately related to the meanings that they attribute to words. When children learn conventional systems of signs, such as natural language and number systems, they need to master two types of meaning: a representational meaning, that connects the signs to something external to the system of signs, and an analytical meaning, that is defined by the system. In the context of natural numbers, children must learn that, as they count, they are adding: the meaning of each number is based on a relation to the previous number $(+1)$ and also based on a relation to its successor (-1). This relational conception of number meanings implies that any number has an infinite number of analytical meanings (e.g. 8 means $7+1$, $6+2$, $5+3$, $9-1$, $10-2$ and so on).

Mastery of the relational meaning of numbers transforms counting. For example, children become able to count on from the number that represents a hidden group of objects and to count on from the value of a coin; learning the relational meaning of natural numbers means that children no longer need to see the items in order to account for them when trying to find the cardinal of a collection or the total amount of money that someone has. Additive relations between units are represented by natural numbers and ratios are represented by rational numbers. Learning conventional numerical systems empowers children's thinking.

3. What are the basic types of relations between quantities that students need to master in primary school?

In primary school, students need to master two types of relations between quantities: part-whole and ratio. Part-whole relations are additive and rely on students' action schemas of joining, separating and setting items in one-to-one correspondence. Ratio relations are multiplicative and rely on the action schema of one-to-many correspondence. Additive and multiplicative reasoning have distinct origins and students can be helped to recognize their difference from an early age by solving both additive and multiplicative reasoning problems in action.

4. How can schools promote students' thinking about relations between quantities?

Research has shown several ways in which schools can promote students' thinking about relations between quantities. The first step is to engage students in thinking in action: modeling a variety of story problems with manipulatives is an important pathway to develop their reasoning about relations between quantities. The next step is to support students' thinking about their problem solving in action. Talking about processes of solving problems enhances metacognition, which in turn promotes further understanding of the problem solving processes. The use of drawings and diagrams is also a pathway towards abstraction. Diagrams aim to capture the essential relations between elements rather than the particularities and support talking about ideas by pointing to aspects of the diagram.

To conclude, it is important to bear in mind that the development of intelligent action based on cultural tools, such as number systems and arithmetic operations, is not an instantaneous process: it takes time and nurturing. Schools are assigned the task and awarded the privilege of promoting students' development, from thinking in action to mathematical models.

References

M. Z. Agustin, M. Z, M. Agustin, P. Brunkow and S. Thomas (2012). Developing quantitative reasoning: Will taking traditional math courses suffice? An empirical study. *The Journal of General Education*, 61(4), 305–313.

K. C. Ang (2001). Teaching mathematical modelling in Singapore schools. *The Mathematics Educator*, 6(1), 63–75.

K. C. Ang (2015). Mathematical modelling in Singapore schools: A framework for instruction. *In Mathematical modelling: From theory to practice*. World Scientific. pp. 57–52

J. Becker (1993). Young children's numerical use of number words: Counting in many-to-one situations. *Developmental Psychology*, 29(3), 458–465.

L. R. Booth (1981). Child-methods in secondary mathematics. *Educational Studies in Mathematics*. Vol. 12, pp. 29–41.

M. Brooks (2009). Drawing, visualisation and young children's exploration of "big ideas". *International Journal of Science Education*, 31(3), 319–341.

M. Brown (1981). Number concepts. In K. Hart (Ed.), *Children's Understanding of Mathematics*: 11–16 (pp. 23–47). Windsor, UK: NFER-Nelson.

S. Carey (2004). Bootstrapping and the origins of concepts. Daedalus, 133, 59-68.

T. P. Carpenter and J. M. Moser (1982). The development of addition and subtraction problem solving. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale (NJ): Lawrence Erlbaum. pp. 10–24

T. P. Carpenter, J. Hiebert and J. M (1981). Problem structure and first grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*. Vol. 12, pp. 27–39.

Y. K. Cheong and M. Consultancy (2002). The model method in Singapore. *The Mathematics Educator*, 6(2), 47–64.

J. Correa, T. Nunes and P. Bryant (1998). Young children's understanding of division: The relationship between division terms in a noncomputational task. *Journal of Educational Psychology*, 90(2), 321–329.

K. Cramer, T. Post and S. Currier (1993). Learning and teaching ratio and proportion: research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics*. New York: Macmillan. pp. 159–178.

C. Csíkos, J. Szitányi and R. Kelemen (2012). The effects of using drawings in developing young children's mathematical word problem solving: A design experiment with third-grade Hungarian students. *Educational Studies in Mathematics, 81*, 47–65.

K. Davidson, K. Eng and D. Barner (2012). Does learning to count involve a semantic induction? *Cognition*, 123(1), 162–173.

L. Ellis (2015). How does the provision of materials in multiplicative reasoning word problems affect year one children's problem solving? An exploration of the impact of representations on children's ability to reason multiplicatively prior to formal instruction. (MSc in Child Development and Education), University of Oxford.

S. Elrod (2014). Quantitative reasoning: The next "across the curriculum" movement. *Peer Review*, 16(3), 4–8.

N. S. Fong and K. Lee (2009). Model method: A visual tool to support algebra word problem solving at the primary level. In *Mathematics education: the Singapore journey*. World Scientific. pp. 169–203.

O. Frydman and P. Bryant (1994). Children's understanding of multiplicative relationships in the construction of quantitative equivalence. *Journal of Experimental Child Psychology*, 58, 489–509.

O. Frydman and P. E. Bryant (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development*, 3, 323–339.

J. Garofalo and F. K. Lester Jr (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education*, 16(3), 163–176.

K. Gläser and P. Riegler (2015). Beginning students may be less capable of proportional reasoning than they appear to be. *Teaching Mathematics and Its Applications*, 34, 26–34. doi:0.1093/teamat/hru025

K. Gravemeijer (1997). Mediating between concrete and abstract. In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective*. Hove (UK): Psychology Press. pp. 315–346.

K. Gravemeijer, M. van den Heuvel and L. Streefland (1990). *Context free production tests and geometry in realistic mathematics education*. *Culembrog*. The Netherlands: Technipress.

D. Guedj (1998). *Numbers. A universal language*. London: Thame and Hudson.

G. Harel and J. Confrey (1994). *The Development of Multiplicative Reasoning in the Learning of Mathematics*. Albany New York: State University of New York Press.

T. Hudson (1983). Correspondences and numerical differences between sets. *Child Development*, 54, 84–90.

D. Hume (1896). *A treatise of human nature*, Clarendon Press.

T. H. Kho, S. M. Yeo and L. Fan (2014). Model method in Singapore primary mathematics textbooks. *Development (ICMT-2014)*, 275.

E. Kornilaki (1999). *Young children's understanding of multiplicative concepts. A psychological approach*. University of London, London.

E. Kornilaki, and T. Nunes (2005). Generalising Principles in spite of Procedural Differences: Children's Understanding of Division. *Cognitive Development*, 20, 388–406.

V. Kouba (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education*, 20, 147–158.

B. Kramarski, Z. R. Mevarech and M. Arami (2002). The effects of metacognitive instruction on solving mathematical authentic tasks. *Educational Studies in Mathematics*, 49, 225–250.

S. Lamon (1994). Ratio and Proportion: Cognitive Foundations in Unitizing and Norming. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics*. Albany, New York: State University of New York press. pp. 89–122.

S. J. Lamon (1993). Ratio and Proportion: Children's Cognitive and Metacognitive Processes. In T. Carpenter P., E. Fennema, & T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research*. Hillsdale, N.J.: Lawrence Erlbaum Associates. pp. 131–156.

N. H. Lee, D. J. S. Yeo and S. E. Hong (2014). A metacognitive-based instruction for Primary Four students to approach non-routine mathematical word problems. *ZDM*, 46(3), 465–480. doi:10.1007/s11858-014-0599-6

E. Mandelbaum (2015). Associationist theories of thought. https://plato.stanford.edu/entries/associationist-thought/.

R. Mayes (2019). Quantitative reasoning and its rôle in interdisciplinarity. *In Interdisciplinary Mathematics Education*. Springer, Cham. pp. 113–133.

J. Mulligan (1992). Children's solutions to multiplication and division word problems: a longitudinal study. *Mathematics Education Research Journal*, 4, 24–41.

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young Children's Intuitive Models of Multiplication and Division. Journal for Research in Mathematics Education, 28, 309–330.

J. Mulligan and M. Mitchelmore (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, 21(2), 33–49.

A. Murata (2008). Mathematics Teaching and Learning as a Mediating Process: The Case of Tape Diagrams. *Mathematical Thinking and Learning*, 10(4), 374–406. doi:10.1080/10986060802291642

P. Nesher (1988). Multiplicative school word problems: theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades*. Hillsdale, NJ: Erlbaum. pp. 19–40.

S. F. Ng and K. Lee (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, 40(3), 282–313.

T. Nunes and P. Bryant (2015a). The development of quantitative reasoning. In L. S. Liben & U. Müller (Eds.), *Handbook of child psychology and developmental science* (7th ed., Vol. 2. Cognitive Process, pp. 715–762). Hoboken, NJ: Wiley.

T. Nunes and P. Bryant (2022a). Using Mathematics to Understand the World. How Culture Promotes Children's Mathematics. London: Taylor & Francis.

T. Nunes and P. Bryant (2022b). Number systems as models of quantitative relations. In G. K. Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education*. Cham, Switzerland Springer Nature Switzerland.

T. Nunes, P. Bryant, R. Barros and K. Sylva (2012). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology*, 82(1), 136–156.

T. Nunes, P. Bryant, D. Burman, D. Bell, D. Evans and D. Hallett (2008a). Deaf Children's Informal Knowledge of Multiplicative Reasoning. *Journal of Deaf Studies and Deaf Education*, 14, 260–277.

T. Nunes, P. Bryant, D. Burman, D. Bell, D. Evans, D. Hallett and L. Montgomery (2008). Deaf children's understanding of inverse relations. In M. Marschark & P. C. Hauser (Eds.), *Deaf Cognition*. Oxford: Oxford University Press. pp. 201–225.

T. Nunes, P. Bryant, D. Evans and D. Bell (2008b). *Children's Informal Knowledge of Multiplicative Reasoning*. Unpublished report.

T. Nunes, P. Bryant, D. Evans, and R. Barros (2015b). Assessing quantitative reasoning in young children. *Mathematics Teaching and Learning*, 17(2-3), 178–196. doi:10.1080/10986065.2015.1016815

T. Nunes, P. Bryant, D. Evans and Bell (2010). The scheme of correspondence and its role in children's mathematics. *British Journal of Educational Psychology*, — Monograph Series II, Number 7 - Understanding number development and difficulties, 83–99.

T. Nunes, P. Bryant, D. Evans, D. Bell, S. Gardner, A. Gardner J. Carraher (2007). The Contribution of Logical Reasoning to the Learning of Mathematics in Primary School. *British Journal of Developmental psychology*, 25, 147–166.

T. Nunes, P. Bryant, D. Evans, L. Gottardis and M. -E. Terlektsi (2014). The cognitive demands of understanding the sample space. *ZDM Mathematics Education*, 46, 437–448. DOI 410.1007/sl 1858-1014-05813.

T. Nunes, P. Bryant, D. Hallett, D. Bell and D. Evans (2009). Teaching Children about the Inverse Relation between Addition and Subtraction. *Mathematical Thinking and Learning*, 11(1), 61–78.

T. Nunes, B. V. Dorneles, P. -J. Lin and E. Rathgeb-Schnierer (2016). Teaching and Learning About Whole Numbers in Primary School. In *Teaching and Learning About Whole Numbers in Primary School*. Cham: Springer International Publishing. pp. 1–50.

T. Nunes, A. D. Schliemann and D. W. Carraher (1993). *Street Mathematics and School Mathematics*. New York: Cambridge University Press.

N. Panorkou and E. F. Germia (2021). Integrating math and science content through covariational reasoning: the case of gravity. *Mathematical Thinking and Learning*, 23(4), 318–343.

J. Park and T. Nunes (2001). The development of the concept of multiplication. *Cognitive Development*, 16, 1–11.

J. Piaget (1952). The Child's Conception of Number. London: Routledge & Kegan Paul.

M. Riley, J. G. Greeno and J. I. Heller (1983). Development of children's problem solving ability in arithmetic. In H. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press. pp. 153–196.

M. Schneider and E. Stern (2009). The inverse relation of addition and subtraction: A knowledge integration perspective. *Mathematical Thinking and Learning*, 11(1-2), 92–101.

A. H. Schoenfeld (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education*. Hillsdale, NJ: Lawrence Erlbaum Ass.

X. She and T. Harrington (2022). Teaching Word-Problem Solving through Tape Diagrams. *Mathematics Teacher: Learning and Teaching PK-12*, 115(3), 170–182.

J. Smith and P. W. Thompson (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the Early Grades.* New York: Erlbaum, pp. 95–132.

S. Squire and P. Bryant (2003). Children’s understanding and misunderstanding of the inverse relation in division. *British Journal of Developmental Psychology*, 21, 507–526.

L. P. Steffe (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259–309.

S. S. Stevens (1946). On the theory of scales of measurement. *Science*, 103, 670–680.

L. Streefland (1997). Charming fractions or fractions being charmed? In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective*. Hove (UK): Psychology Press. pp. 347–372.

P. W. Thompson (1993). Quantitative Reasoning, Complexity, and Additive Structures. *Educational Studies in Mathematics*, 3, 165–208.

P. W. Thompson (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlin & L. L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education*. Laramie, WY, USA: University of Wyoming. pp. 33–57.

P. W. Thompson and L. A. Saldanha (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics*. Reston, *VA: National Council of Teachers of Mathematics*. pp. 95–113.

R. C. Tolman (1917). The measurable quantities of physics. *Physical Review*, 9(3), 237–253.

G. Vergnaud (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*. London: Academic Press. pp. 128–175.

G. Vergnaud (2009). The theory of conceptual fields. *Human Development*, 52, 83–94.

H. Von Helmholtz (1921). Epistemological Writings (Vol. XXXVII). Dordrecht/Boston: Reidel.

A. Watanabe, T. Nunes, P. Bryant and M. van den Heuvel-Panhuizen (2000). An assessment of mathematics concept development for primary school. Paper presented at the BPS Developmental Annual Meeting, Nottingham.

W. K. Yoong, L. P. Yee, B. Kaur, F. P. Yee and N. S. Fong (2009). *Mathematics Education: The Singapore Journey*. Singapore World Scientific.